SCIENCE

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MSS. intended for publication and books, etc., intended for review should be sent to Professor J. McKeen Cattell, Garrison-On-Hudson, N. Y.

THE PROVISION MADE BY MATHEMATICS FOR THE NEEDS OF SCIENCE[1]

MATHEMATICS beyond the merest elements has been regarded by some as an excrescent malady of the human spirit, generated like the pearl in an abnormal and morbid way and representing a non-living embedment in the active tissue of the organism of society; by others it has been supposed to exhibit the highest intellectual reach of mankind, being in itself the most powerful tool yet devised for the interpretation of natural phenomena, while at the same time it affords a satisfying expression of the furthermost esthetic attainment. On the one hand, it is considered a piece of jugglery in which it is the joy of the proficient to produce more and more complicated entanglements to astonish the beholder and overwhelm him with the sense of mystery; on the other hand, it is seen to be the systematic unfolding of remarkable and important properties of a highly fascinating creation or construction of the human spirit by means of which it has at once its most intellectual delight and the best means of understanding its environment. Some workers seem to resent the interference of mathematics with their comfort in the conclusions of descriptive science and its demands that observation shall be reduced to measurable elements and the laws of nature be expressed in mathematical formulas; other thinkers believe that natural science is real science only in so far as it is mathematical, that it is only through mathematics that true science can

[1] An address delivered before the Illinois Chapter of Sigma Xi on January 17, 1917.

be understood, and that without mathematics no science can develop to maturity. One delights in observation and the record of facts, believing that he has understood a class of phenomena when he has given a general description of their relations, order and connections; the other considers the mathematical formula as "the point through which all the light gained by science must pass in order to be of use in practise."

But it is true, I believe, that mathematics is generally recognized as essential at least to scientific progress; and what I intend to do this evening is to discuss the nature of the provision made by it for the needs of science. But I can not consent to enter upon a treatment of this topic without pausing a moment, first of all, to combat a certain dangerous and apparently rather widespread error among scientists, namely, that the primary and perhaps the sole function of mathematics is to assist in the solution of the problems of science. If this were the essential purpose of mathematics, you would not find it cultivated as it is to-day. Instead of this the men who now give all their energy to its development would be allied with the particular natural sciences and would study mathematics only as auxiliary to their central concern. One could find no incentive to labor otherwise.

Fundamentally mathematics is a free science. The range of its possible topics appears to be unlimited; and the choice from these of those actually to be studied depends solely on considerations of interest and beauty. It is true that interest has often been, and is to-day as much as ever, prompted in a large measure by the problems actually arising in natural science, and to the latter mathematics owes a debt only to be paid by essential contributions to the interpretation of phenomena. But after all, the fundamental motive to its activity is in itself and must remain there if its progress is to continue.

Some mathematicians are glad when their fields of thought touch other sciences (or even practical matters); others concede the fact unwillingly or without interest; others still would perhaps consider themselves unorthodox in their feeling if they allowed any matter of connections with other things to affect at all their interest in their own fields. It was perhaps an extreme case of this feeling which prompted Sylvester to the pious or impetuous hope that no use would ever be found for the theory of invariants which he was developing with so much delight.

But it has turned out, as the mathematician is now well aware, that it is these same invariants which afford us an expression for the laws of nature. An invariant is simply a thing or a relation which remains unaltered when the elements with which it is connected undergo a certain class or group of transformations. When we know the transformations to which a class of phenomena are subject the matter of finding out the laws connecting these phenomena is a problem of invariants. It may be that a particular physical law was discovered long before the idea of invariants arose; but it is nevertheless true that a useful connection among such laws is afforded by this notion. On the other hand, we may have an equation expressing a fundamental relation among a large class of phenomena and find that this equation is invariant when the elements in it are subjected to a certain group of transformations. We may be sure that this group of transformations has something essential to do with the phenomena in consideration, and that its invariants express (partially or completely) the laws governing these phenomena. Our problem

is then to deduce these invariants and to give to them a physical interpretation.

In the theory of invariants we have an illustration of a fundamental fact concerning the applications of any but the most elementary mathematics, namely, that they arise essentially as by-products of the leading discoveries. On the part of those who make use of them they are often considered as the essential and perhaps the only results of mathematical investigation. There are some who have been impatient of those studies which apparently have no connection with less abstract considerations. But this is a short-sighted impatience. He who wishes applications and applications alone can not secure his ends better than by the encouragement of theoretical investigations even of an abstruse and remote character. Within the separate natural sciences this is so well understood that it is a matter of surprise to observe that some individuals persistently refuse to carry the conception to its logical consequence as regards the science of mathematics. But any one who has meditated much on the character of the progress of thought must certainly have a profound realization of its truth.

It seems that no body of thought has been of more importance in human progress and at the same time been criticized more freely than the science of mathematics. Much of this criticism appears to be good-natured and to amount to but little more than a quasi-humorous way of expressing the critic's own unashamed ignorance. At first sight one might treat this as harmless; but from the point of view of the general interest it can hardly be passed over in such a way. How this ignorance is to be overcome I can not say. Perhaps one of the first requisites is to find some means of overcoming the shamelessness with which individuals otherwise well trained contemplate their own ignorance of mathematics.

The mathematician himself is not disturbed so far as the welfare of his own science is concerned; but it is sometimes a matter of pain to see the general loss which arises from such ignorance and also from the severer strictures of the more pronounced adversaries of mathematics. In no other case, however, have the criticisms been so severe as those meted out to the infinitesimal calculus in the infancy of its development; and never have the fondest hopes of the founders of a science been so far surpassed by its actual achievements as here where the subject has become central in practically every field of pure and applied mathematics.

It will be instructive to examine briefly the criticisms thus met so early by the infinitesimal calculus. Some persons attacked the certainty of its principles, attempting to show that its conclusions were at variance with those obtained by methods previously known and accepted as sound. Some who labored primarily with matters of morality and religion attacked the new departure of thought on general grounds; they repulsed themselves by unwittingly displaying their ignorance of the thing which they criticized. One man, who entrenched himself in masses of calculation, pronounced the procedure of the new calculus unsatisfactory because of the indeterminacy of the form in which certain results appeared; but he afterwards acknowledged his error and admitted that he had been urged forward by malevolent persons—a thing (let us believe) which does not often happen among workers in science. Christiaan Huygens, whose opinion probably carried more weight than that of any other scientific man in his day, believed that the employment of differentials was unnecessary and declared that Leib-

nitz's second differential was meaningless. But these and many other criticisms never hindered the development of the new calculus, but served rather to aid in clearing off certain excrescences which had nothing to do with its essential characteristics and in helping it to that central place of importance which it holds to-day.

The criticism last mentioned is one which is made so often that it is profitable to dwell longer upon it. So often the mathematician hears: "What is the use of what you are doing?" He knows a thousand answers to this question; and one of the most effective is that which history has given to the criticism of the illustrious Huygens. The recently developed subject of integral equations has sometimes been confronted with the inquiry: Why develop this theory? Will not differential equations serve the purpose? But the mathematician goes calmly ahead with the development of those things which interest him just as he did formerly; and in the new case he anticipates with confidence the same triumphant justification in the event which has uniformly crowned his labors in the past.

Sometimes the criticisms directed against mathematics have grown out of a misconception of the natural limitations to which it is subject. Pages of formulas can not get a safe result from loose data. No amount of computation will remove from a result the errors already existent in the underlying observations. I have several times been confronted with the statement that mathematics made a great mistake in this or that particular case in predicting what was not found on proper examination to be true. But after all there was nothing wrong with the mathematics. It was merely that the supposed laws of phenomena on which the investigation was based were not exact. It was they and not the mathematics which were on trial.

It is true, as one of my friends said to me recently, that no machine can be constructed and completely theorized on mathematical principles alone. When this is given as a statement of fact I have nothing to say in reply. But if this natural limitation of the subject is spoken of (as I have sometimes heard) as an expression of the failure of mathematics, then an error is made which ought to be corrected. Mathematics does not claim to do the whole thing in the development of science. It simply has its rôle to perform; and there is a devoted body of workers throughout the world striving to see that it perform this function with eminent success. How it does and is to continue doing this will be apparent from the following discussion of the relation of mathematics to experimental verification.

By their very nature the conclusions of pure mathematics are not subject to experimental examination. One would not say that they are above or beyond experience, but that they are outside of it. Pure mathematics deals with certain creations of the human spirit and with these alone. So far as it is concerned, no import attaches to the inquiry after the impulse which resulted in the creation of these things. The mathematician *qua* mathematician is not interested in this matter, however much it may fascinate him as a philosopher; and he develops his science usually with perfect indifference to such considerations, rearing it from a small group of postulates or perhaps even from the general logical premises of all reasoning. In any event, he drags out into the limelight all hypotheses and keeps them vividly before him during the progress of his investigations.

In applied mathematics the state of things is very different. Here the whole treatment is bristling with implicit assumptions, some of them being carried con-

sciously while many of them are apparently unperceived. In other words, while we are applying mathematics we are at the same time making use of the customary large bundle of prejudices and preconceptions which we have not yet found a way to avoid whenever we have to treat the phenomena of nature.

In order to illustrate the way in which applied mathematics is shot through with assumptions, let us take a single case. When we begin to apply numbers in the measurement of physical quantities we assume at once that the true measure may as well be an irrational number as a rational number. For this assumption we have no *a priori* grounds or experimental reasons. We simply find it the most convenient assumption to make and we make it, having no other justification than convenience for our procedure. It may very well be true that the universe is so constructed that every measurement in it yields essentially a rational number. This would be true if all material things, all force, all energy, etc., were granular in structure and the mutual ratios of all granules were commensurable. For instance, if it should turn out that mass is to be estimated by counting the number of granules (all alike) in a body, then the mass of the body would be expressed essentially by a rational number. If electricity consists altogether of electrons of equal charge, then the measure of any charge of electricity is essentially a rational quantity. But we treat the phenomena mathematically as if essentially irrational quantities occur generally in nature. Even the continuity of space apparently rests upon just such sheer assumptions.

Starting with this fundamental hypothesis of applied mathematics, we might follow the subject from these remotely abstract regions into the things of more common thought, and in doing so we should find that such fundamental assumptions obtrude themselves at every point. If we looked persistently only at this side of the matter we should probably lose all confidence in our theoretical interpretations; but fortunately we are able always to test our results approximately with experimental data. It is not that we are testing out our processes of reasoning (we have another method for doing that); but rather that we are examining as to whether we have found a construction and interpretation which fit in with phenomena to a satisfying degree.

After these remarks I will hardly need to urge the necessity of testing in the laboratory all results obtained from given hypotheses by logical processes. For we can never know the truth of any hypothesis, or even understand its import, until we know the consequences which flow from it. Whether the conclusions reached in this be determined only by an appeal directed to experience.

Among the fields of applied mathematics that of rational mechanics has been most completely transfused by the mathematical spirit and it is here that the latter has exhibited some of its most characteristic conquests. It has here shown how high mathematical skill on the part of some investigators is necessary to the greatest progress of science, illustrating the way in which the mathematical spirit and method furnish a bond of union to the separate divisions of physical science.

So far we have considered the character of the provisions made by mathematics for the needs of science. It remains to give some specific details as to the past and some indication of apparently probable lines of development in the future. Clearly, a catalogue of specific provisions would be impossible in this address; such a thing

would require not less than a year's course of lectures. We can not hope to do more than indicate some of the central provisions.

Let us begin with a consideration of the early stages in the development of particular sciences. Each separate experimental science passes through a period of infancy in which it is not able to stand the strong meat of mathematics; and mathematical ideas initially find an application in it by slow processes. The first essential is to gather data—descriptive results, measured quantities, or what not; and only after much labor does law become apparent and the mathematical tool acquire its characteristic power.

But even in these preliminary stages mathematics makes an essential contribution in a preliminary way. It furnishes the only language in which exact information can be expressed, recorded and conveyed; and in this respect is a necessary element of that collaboration which is essential to such rapid progress as has been usual in recent years. But it does more than this. It enables us to record observations in such a way that we are able readily to grasp the relations of the various measured elements involved. I refer here particularly to the use of graphs which present data in so compact a manner and in a way so well adapted to our intuitive realization of their significance.

It would now probably be impossible to lay the foundations of any new experimental science without the collection of much numerical data, that is to say, without the use of statistics. But how are these to be interpreted? Clearly it must be by the methods of statistical mathematics.

Let us suppose now that we have made a record of our measurements of phenomena, their juxtapositions, their magnitudes, their order in time; let us assume (as we always do and must) that they are connected by law. How shall we ascertain what that law is? By what criterion shall we judge of the accuracy of our hypothetical explanation? Certainly not on any absolute grounds; we can only select the explanation which seems to us most probable. And for this our best and surest guide is and must be the mathematical theory of probability.

A science in the stage now being examined would properly be called non-mathematical, notwithstanding the preliminary use which it makes of mathematical science. Among those divisions of systematic thought which are at present in this stage of development one would probably include political science, economics, biology, psychology and geology.

At the next stage of development one would think of the individual science as having come to the period of vigorous youth but not yet mature. As preeminently the example of such a science I would select chemistry. By no means is it reduced essentially to mathematical form; and yet its laws are so stated as to be subject to the sharpest experimental verification. It employs the mathematical tools which are common in the earlier stages of science and also some additional ones. It may even employ the first derivative as a measure of rate of the reactions which it considers; but I believe that it seldom or never makes use of the second derivative.

It may be taken as a mark of the more advanced development of physics that it finds constant use for derivatives of the second order and sometimes for those of higher orders. Some one has expressed this increasing order of complexity and certain accompanying dependencies by saying that behind the artisan is the chemist, behind the chemist is the physicist, and behind the physicist is the mathematician—a pleasing climax, at least if you are a mathematician.

When we look upon physics, for instance,

as a mature science, we are not to think of it as having become dead and unproductive. Like the individual scientific worker on coming to maturity, it has merely reached the period when its deeds become most effective for the use and satisfaction of mankind. In fact, physics is perhaps at the same time the most mature natural science at present existing and the one whose recent progress has been the most rapid and the most remarkable.

In the development of physics the infinitesimal calculus has persistently played a leading rôle; its interaction with experimental results has been and is fundamental and necessary to the progress we have witnessed and yet see to-day. From this creation of the mathematicians and the use made of it by the physicists the world has received a good practically immeasurable in its extent. Sometimes we are tempted to assess the advantages due to each of these elements; but one can hardly expect success from such a venture. Logically the mathematics is prior; for it could exist of itself, while the physics probably could not. But psychologically and practically they are so bound up that no separation can be made. Were the mathematics swept away, much of physical theory would likewise have to go; but on the other hand much of the mathematics would never have existed had it not been called into being by the demands of physical science.

Until recently it was customary to assume that nature is essentially continuous in her manifestations. As long as we proceed on that hypothesis the infinitesimal calculus is the natural tool to be employed in the investigation of phenomena; and we should expect to find differential equations and integral equations playing a leading rôle in the exposition of physical theory.

That they have done so has furnished a great incentive to some investigators in prosecuting their labors. It is said that Poincaré was urged on in his studies of differential equations by the conviction that he was engaged in perfecting the most important tool which could be employed in the investigation of physical phenomena. No doubt it is a similar use of integral equations which drew quickly into that field so large a body of workers and resulted in its so rapid development. The same spur has urged men on in the study of expansion problems in connection with both differential equations and integral equations. It is now a long while since Fourier series were thus introduced; and their properties and availability have been treated in numerous investigations.

More recently extensive generalizations of these series have made their appearance; and we have a great class of expansions in the so-called orthogonal and biorthogonal functions arising in the study of differential and integral equations. In the field of differential equations the most important class of these functions was first defined in a general and explicit manner by an American mathematician, Professor Birkhoff, of Harvard University; and their leading fundamental properties were developed by him. We shall doubtless witness a great progress in our knowledge of these functions.

But in the early years of the present century the world of scientific thought has been unexpectedly confronted with a new situation of a rather astonishing sort. Our unquestioning assumption of the continuity of nature appears now not to have been well founded; and much of the development of theory which has been based on it is consequently perhaps to be regarded as only a rough and unsatisfactory first approximation. If certain apparent discontinuities in nature turn out to be real (and it looks now as if they must) then the differential equation will probably lose its place as the most important tool of applied mathematics

and the corresponding expansions will no longer serve to yield us the most satisfying form for the expression of our results.

This situation has been contemplated with uneasiness in certain quarters. To some natural scientists it has seemed like the loss of our moorings. To some mathematicians it has appeared in the light of greatly lessening the importance of many investigations, difficult and prolonged. It is said that Poincaré contemplated the outlook with keen regret. But we had as well make up our minds to the situation. It seems almost certain that electricity is done up in pellets, to which we have given the name of electrons. That heat comes in quanta also seems probable. In fact, it is not unlikely that we are on the verge of interpreting everything in nature as essentially discontinuous; and it would perhaps be no surprise to us now to find that even energy itself is not unlimitedly divisible, but exists, so to speak, in granules which can not be separated into component parts.

A few years ago such a paragraph as the foregoing would have been thought a piece of nonsense and to be not entitled to consideration; now the author is more likely to be charged with repeating something which already has been heard to the point of weariness.

In view of so sweeping and fundamental changes in our outlook, what is going to happen to the existent body of applied mathematics? Simply this, if these new ideas gain currency: that which before had been considered a very close approximation to facts will now be treated as giving only a coarse first approximation; and we shall set about the task of finding means of studying phenomena more exactly in consonance with the new underlying ideas.

You will probably be disposed to ask in what direction we shall turn now to find the requisite mathematical tools and when we can expect to have them ready for use. It may be answered that the mathematician was beforehand with a partially developed tool which will probably serve the purpose. When these new ideas in physics were just coming to the front a few young mathematicians independently of each other and apparently without knowledge of these movements in physics were engaged in the study of certain mathematical problems having to do with a thing which will probably turn out to be a suitable tool for the investigation of discrete phenomena. At any rate, the equations which they were studying are not intimately bound up with considerations of continuity as are differential equations, but yet they possess a number of properties very similar to or in common with those of differential equations. The equations which are thus brought to a new position of importance are the so-called difference equations.

Simple difference equations first appeared in the literature rather early in the history of mathematics and certain elementary aspects of their theory were considered several generations ago. But in recent years an essentially new type of problem in connection with them has come to notice; and in a short time and through several independent investigators the theory has suddenly blossomed forth into unexpected and magnificent flower.

This development had its origin almost simultaneously in three countries and in the hands of three independent investigators: Nörlund in Sweden, Galbrun in France, and myself in America. My own first contribution was followed closely (also in this country) by Birkhoff's first fundamental paper in the new field. By this time numerous other persons have made contributions to the development of the subject both from the function-theoretic point of view of the papers just mentioned

and also from another direction with a consequent consideration of a different type of problem.

So far the recent workers in this field have given their entire attention to purely mathematical developments and have not considered so much as the possibility of a use of the results in the domain of applied mathematics. In particular, my own interests have been in theoretical considerations. But I look forward to important applications of this newly developed theory both because it seems to have in it some at least of those elements which are necessary to accord with phenomena which are discontinuous in their nature and more particularly because there is here an expansion theory also consonant with the discontinuities of nature and related to difference equations in a manner somewhat similar to that in which certain other expansion problems attach themselves to differential equations. But the analogy must not be pressed too far, since there are also essential differences.

Concerning these new expansion problems I wish to say one further word. It is very recently indeed that they have come to notice; and a knowledge of them is not yet generally current. In fact, the general definition of the series involved was first made in a paper of my own published less than a year ago; and I am still engaged in working out their more detailed theory. Doubtless other workers in the near future will take up different phases of the same problem.

So far no exposition of the modern theory of difference equations exists in the literature; the results are to be found only in the original memoirs. In a few instances this theory has furnished the basis or an integral part of a course of academic lectures. As such it has appeared, as I understand, in one of the courses at Harvard. I have myself delivered lectures on it in Indiana University and the University of Chicago; and it is my purpose next year to expound this new doctrine in my lectures here. It is highly desirable that this matter shall be developed rapidly and be prosecuted from various points of view. It is in this way only that we shall be able to learn what its import really will be for the progress of science.

Before concluding my remarks I wish to speak briefly of a different sort of conception or expectation which has arisen in some quarters and having to do with a more fundamental and far-reaching use of mathematics than any yet made. It is connected with the fact that every branch of physics gives rise to an application of mathematics and the consequent feeling that there must be a deep underlying reason for this and a consequent close relation of phenomena which probably makes them capable of an explanation from a single point of view consistently maintained.

If there is a "hypothetical substructure of the universe, uniform under all the diverse phenomena," it would appear that there must be some means of ascertaining what it is and of giving to it a mathematical expression and body. At any rate the expectation of such a thing has arisen; let us hope that the event will show that the anticipation is well grounded in the nature of things.

I understand that the earliest contributions to just such a development are already in existence; that the now current theoretical accounts of radiation, diffusion, capillary action and molecular behavior in general have just such characteristics as one would expect to find in the early stages of a mathematical theory of the substructure of the universe.

Let me guard against a misapprehension concerning the foregoing remarks. I have

not been discussing the general elements in scientific progress; my purpose has been much less ambitious. In speaking to my colleagues in other fields I have tried to give an account of the faith that is in me so that they shall see what sort of motives (aside from those of esthetic delight, which however are central) the mathematician has in the work which he pursues. With this purpose before me I have spoken of just one side of the fundamental requisites of scientific progress. And now I wish to say with emphasis that I have the keenest appreciation of the use and purport of other methods and the sharpest delight in the contemplation of their achievements——achievements so different from any to be wrought out by my own familiar and loved mathematical tools. One could hardly speak of these other methods without becoming eloquent in his admiration of them. They are left unmentioned, then, not because I do not appreciate them, but because they do not fall within the scope assigned to this discussion.

My purpose will have been served if I have tended to produce in your minds a keener appreciation of the place of mathematics in the development of scientific thought; and particularly if I have induced in you a conception and feeling of the consecration with which choice mathematical spirits devote their energies to penetrating into the unknown regions of their own creations and to opening up larger areas of the enormously expanding field of mathematics which has grown more in the present generation perhaps than in any other in the history of the world.

R. D. Carmichael

SCIENTIFIC EVENTS

THE GAUTHIOT MEMORIAL

Dr. Robert Gauthiot, directeur d'études adjoint in the Ecole des Hautes Etudes, one of the most brilliant Oriental scholars of our time, died in Paris on September 11, 1916, at the age of forty, from the effects of a wound received as captain of infantry while gallantly leading his company to a grand attack. Gauthiot was a real genius, and has made lasting contributions to Iranian and Indo-European philology, playing a prominent part in the recent movement of opening up the history of Central Asia. To his ingenuity and acumen is due the complete decipherment of the Sogdian, an Iranian language preserved in ancient manuscripts which some years ago were discovered in Turkestan. He conducted two highly successful expeditions into the Pamir for linguistic exploration. Hardly had he reached the Pamir for the second time in July, 1914, when news of the outbreak of the war determined him to return to France and to take his place in the defense of his country, distinguishing himself by his bravery and receiving the croix de guerre.

The loss caused to science by his premature and tragical death is irreparable. He has left in straitened circumstances a widow and four daughters, the youngest being three years of age. A committee has been organized for the purpose of raising a Gauthiot Memorial Fund in commemoration of the great scholar, this fund to be utilized for the maintenance of his destitute family and for the publication of a Gauthiot Memorial Volume. Any further information, if desired, will be gladly given by the secretary. Contributions which will be gratefully acknowledged may be sent to some member of the American committee, or if preferred, directly by draft on Paris to Professor A. Meillet (65 rue d'Alésia, Paris XIV^e, France), treasurer of the French Board of Trustees for the Gauthiot Memorial Fund.

The American committee consists of:

Martin A. Ryerson, 134 South La Salle Street, Chicago—*Honorary President.*

A. V. Williams Jackson, professor of Iranian and Sanskrit, Columbia University, New York.

James H. Breasted, professor of Egyptology and Oriental History, University of Chicago.

Walter E. Clark, professor of Sanskrit, University of Chicago.